Basic Fourier Definitions

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Continuous Periodic Functions

Fourier Series. This can be used to represent any continuous periodic function, and in effect gives the Fourier Transform of continuous periodic functions. The Fourier series itself can be thought of as the inverse transform, whereas the coefficients a_n and b_n represent the forward transform because they give the amplitudes of the various frequency components.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \,\mathrm{Cos}[2\pi f_n t] + b_n \,\mathrm{Sin}[2\pi f_n t]\right)$$

where

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{Cos}[2\pi f_n t]\,dt \qquad \{n = 0, 1, 2, \ldots\} \quad \text{and}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{Sin}[2\pi f_n t]\,dt \qquad \{n = 1, 2, \ldots\}$$

and $f_n = n\, f_0 = n/T$ (T = period; f_0 = fundamental frequency).

Another representation of the Fourier Series (essentially the same thing as above, **but admitting negative frequencies**) is:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{+i\, 2\pi f_n t} \qquad (\mathrm{Inverse}) \quad \{n = \ldots -1, 0, 1, \ldots\}$$

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, e^{-i\, 2\pi f_n t}\,dt \qquad (\mathrm{Forward}) \quad \{n = \ldots -1, 0, 1, \ldots\}$$

Note that the amplitudes of the C_n terms are half those calculated using the first definition for a_n and b_n because the C_n terms exist for both positive and negative n. These last two expressions are in a form that more closely resembles the inverse and forward Fourier Transforms below. Because the function f(t) is periodic, the frequencies are discrete (i.e., just points, not continuous functions). Continuous periodic functions transform into discrete non-periodic functions.

Discrete Periodic Functions (Number Series)

Discrete Fourier Transform. This can be used to give the forward Fourier Transform of the discrete periodic time series g_k or the inverse Fourier Transform of the discrete periodic frequency series G_n. The coefficients G_n represent the forward transform of the time series g_k and give the amplitudes of the discrete frequencies contained in g_k. The forward transform of a time series is a frequency series that repeats with a period of $\frac{1}{\Delta t}$ where Δt is the sample interval in time. Note that N samples represent the function in both the time and frequency domains, and that discrete periodic functions transform to discrete periodic functions.

$$G_n = \frac{1}{N} \sum_{k=0}^{N-1} g_k \, e^{\frac{-i 2 \pi n k}{N}} \qquad \text{(Forward)} \qquad \left(\text{Period} = \frac{1}{\Delta t} = N\right)$$

$$g_k = \sum_{n=0}^{N-1} G_n \, e^{\frac{+i 2 \pi n k}{N}} \qquad \text{(Inverse)} \qquad \left(\text{Period} = \frac{1}{\Delta f} = N\right)$$

Continuous, Non-Periodic Functions

General Fourier Transform. This is used to give the Fourier Transform of continuous functions that are non-periodic (never repeat). These expressions constitute a Fourier Transform pair, and give the forward and inverse transformations to represent the continuous frequency or time functions in the other domain. Continuous non-periodic functions transform to continuous non-periodic functions.

$$F(f) = \int_{-\infty}^{\infty} f(t) \, e^{-i 2 \pi f t} \, dt \qquad \text{(Forward)}$$

$$f(t) = \int_{-\infty}^{\infty} F(f) \, e^{+i 2 \pi f t} \, df \qquad \text{(Inverse)}$$